Solutions of Schlesinger system and Ernst equation in terms of theta-functions

D. A. Korotkin[1]

Max-Planck-Institut für Gravitationsphysik,
Schlaatzweg 1, D-14473 Potsdam, Germany

and

V. B. Matveev [2]

Physique Mathématique, UFR Sciences et Techniques,
Université de Bourgogne, B.P. 138, 21004 Dijon Cedex, France

Abstract

We discuss the relationship between Schlesinger system and stationary axisymmetric Einstein's equation on the level of algebro-geometric solutions. In particular, we calculate all metric coefficients corresponding to solutions of Ernst equation in terms of theta-functions constructed in [20, 21, 25].

Mathematics Subject Classification (1991):

[1]E-mail: korotkin@aei-potsdam.mpg.de
[2]E-mail: matveev@u-bourgogne.fr

1 Introduction

Interaction between algebraic geometry of the compact Riemann surfaces and the theory of integrable systems represent nowadays a well established paradigma of modern mathematical physics. The analysis on the compact Riemann surfaces was essentially completed in 19th century in the famous classical works by Gauss, Euler, Riemann, Jacobi, Weierstrass etc. The penetration of the tools provided by their methods in the theory of integrable systems started in the same time. Legrange first solved the equations of motion of the Euler top by means of the use of the Jacobi elliptic functions. Next essential breakthrough was achieved in 1889 by Kovalevski [1] who found the new integrable case of the motion of rigid body solvable in terms of genus 2 algebraic functions. Slightly later Dobriner published the explicit solution of sine-Gordon equation expressed in terms of two-dimensional theta-functions [2]. Then Carl Neumann solved the equations of geodesic motion on three-axis ellipsoid in terms of two-dimensional hyperelliptic theta-functions [3].

The same time it started the development of the spectral theory of the Sturm-Liouville operators with periodic coefficients. First important steps in the field were done by Floke [4], Lyapunov [5] and others (see related references in the book [6].

In 1919th french matematician Jule Drach wrote the remarkable article (absolutely forgotten for next 60 years) devoted to the construction of explicitly solvable Sturm-Liouville equations associated to hyperelliptic curves of any genus [7].

One of the subsequent developments of the discovery of the inverse scattering transform (IST) method by Gardner, Green, Kruskal and Miura [8] was the complete understanding of the deep interplay between the classical works listed above. This connection now became a part of what is called algebrogeometric approach to the solution of nonlinear differential equations. This field of activity was initiated in 1974-1976 by the works of (in chronological order) Novikov, Lax, M.Kac, Its, Matveev, Dubrovin, McKean, van Moerbeke and Krichever.

Speaking about the aspects of the theory connected to the explicit solutions of these equations one should mention the formulas for solutions of the Korteveg-deVries equation with periodic initial data obtained in [9]. Soon after the same kind of formulas were derived for numerous other equations integrable via IST method, including Non-linear Schrödinger [10], Sine-Gordon [11], Toda lattice [12], Kadomtzev-Petviashvili [13] and so on. The generic feature of the algebro-geometric solutions of the systems listed above is that they are parametrised by the fixed algebraic curves and the associated dynamics is linear on their Jacobians. These results found many beautiful and unexpected applications in various branches of modern mathematics and physics, including differential geometry of surfaces [14] and algebraic geometry (Novikov hypothesis [15]).

It is also relevant to notice that simultaneously with the soliton theory, the algebro-geometric methods were actively applied in the framework of twistor theory (which, essentially, also deals with integrable equations, and exploits the methods very closely related to ones used by integrable systems community, but essentially focuses on the global properties of the solutions). Probably, the main achievments in this direction were classifications of the instanton and self-dual monopole configurations (see for more details the book [16] and references therein).

Technically and conceptually new development was the application of algebrogeometric ideology to the Einstein equations of general relativity in presence of two commuting Killing vectors (Ernst equation). The embedding of this equation in the framework of the

IST approach was initiated by Belinskii-Zakharov [17] and Maison [18] [3]. The characteristic feature of the associated zero-curvature representation is the non-trivial dependence of the associated connection on the spectral parameter. Namely, the connection lives on the genus zero algebraic curve depending on space-time variables. This peculiarity entails the drastic change of the construction and qualitative properties of algebro-geometric solutions first obtained in 1988 [20, 21]. Dynamics in these solutions is generated by hyperelliptic curves with two coordinate-dependent branch points. The algebro-geometric solutions of the Ernst equation do not possess any periodicity properties which were inevitable for all KdV-like cases studied before. Moreover, we can explicitly incorporate in the construction an arbitrary functional parameter which never appears in the traditional KdV-like 1+1 integrable systems. Wide subclass of the obtained solutions turns out to be asymptotically flat [22]. As a simple degenerate case the algebro-geometric solutions contain the whole class of multisoliton solutions found by Belinskii and Zakharov [17].

Simplest elliptic solutions were studied in [22]. Despite many interesting properties, they contain ring-like naked singularities making it difficult to exploit them in a real physical context. More realistic physical application of the particular family of the class of algebro-geometric solutions came out from the series of papers of Meinel, Neugebauer and their collaborators starting from 1993 [23, 24]. These works were devoted to the investigation of the boundary-value problem corresponding to the infinitely thin rigid relativistically-rotating dust disc. The explicit embedding of the dust disc solution in the formulas of [20] was given in [25]. In the subsequent series of papers Klein and Richter (see, for example, [26, 27]) established the link of the algebrogeometric solutions of Ernst equation given in [20, 25] with the scalar Riemann-Hilbert problem on hyperelliptic curve and further discussed potential applications to rotating bodies.

Purpose of this paper is twofold. First, we extend the link between isomonodromic solutions of the Ernst equation and classical Schlesinger system [28], established in the paper [30], on the level of algebro-geometric solutions. Namely, we show how all algebro-geometric solutions of the Ernst equation may be obtained from the algebro-geometric solutions of the Schlesinger system found in the recent paper [29]. This allowes to get remarkably short expression for algebro-geometric solutions of the Ernst equation:

$$\mathcal{E}(\xi,\bar{\xi}) = \frac{\Theta\begin{bmatrix}\mathbf{r}\\ \mathbf{s}\end{bmatrix}\left(V|_{\xi}^{\infty^1}\right)}{\Theta\begin{bmatrix}\mathbf{r}\\ \mathbf{s}\end{bmatrix}\left(V|_{\xi}^{\infty^2}\right)}\ , \tag{1.1}$$

in terms of theta-functions with constant characteristics $\mathbf{r},\mathbf{s}\in\mathbb{R}^{g_0}$, associated to hyperelliptic curve $\mathcal{L}_0$ of genus g_0 with two coordinate-dependent branch points:

$$\nu^2 = (w-\xi)(w-\bar{\xi})\prod_{j=1}^{2g_0}(w-w_j)\ . \tag{1.2}$$

Applying certain limiting procedure to this solution, we can get general algebro-geometric solution of the Ernst equation.

Second, we give explicit expressions for all metric coefficients corresponding to algebro-geometric solutions of the Ernst equation. The most non-trivial part is the calculation

[3]Coming back to the works of classics of differential geometry of 19th century it is curious to mention that essentially the same Lax pair appeared in the work of Bianchi [19] in the study of so-called Bianchi congruences.

of so-called conformal factor; for this we use the link between the conformal factor and tau-function of the Schlesinger system, established in [30], and the formula for the tau-function obtained in [29]. The final formula for the conformal factor corresponding to solution (1.1) looks as follows:

$$e^{2k} = \frac{\Theta\left[\begin{smallmatrix}\mathbf{p}\\ \mathbf{q}\end{smallmatrix}\right](0|\mathbf{B})}{\sqrt{\det \mathcal{A}_0}} \prod_{j=1}^{2g_0} |w_j - \xi|^{-1/4} \,, \tag{1.3}$$

where $\mathcal{A}_0$ - is the matrix of a-periods of holomrphic differentials $w^j dw/\nu$, $j = 1, \dots, g_0$ on $\mathcal{L}_0$, and the theta-function is associated to the hyperelliptic curve $\mathcal{L}$ of genus $2g_0 - 1$ defined by the equation

$$y^2 = \prod_{j=1}^{2g+2} (\gamma - \gamma_j) \,,$$

where

$$\gamma_j = \frac{2}{\xi - \bar{\xi}} \Big\{ w - \frac{\xi + \bar{\xi}}{2} + \sqrt{(w - \xi)(w - \bar{\xi})} \Big\} \,.$$

Appropriate limiting procedure allows to deduce from this formula the expression for conformal factor corresponding to general algebro-geometric solutions. Notice, that it seems to be impossible to give simple expression for the conformal factor entirely in terms of the objects corresponding to curve $\mathcal{L}_0$, which was the obstacle for deriving this formula without understanding the link between the conformal factor and the tau-function.

2 Schlesinger system and stationary axisymmetric Einstein equations

2.1 Schlesinger system

Consider the following linear differential equation for function $\Psi(\lambda) \in SL(2, \mathbb{C})$:

$$\frac{d\Psi}{d\gamma} = A(\gamma)\Psi \tag{2.1}$$

where

$$A(\gamma) = \sum_{j=1}^{N} \frac{A_j}{\gamma - \gamma_j} \tag{2.2}$$

and matrices $A_j \in sl(2, \mathbb{C})$ are independent of γ. Let us impose the initial condition

$$\Psi(\gamma = \infty) = I \tag{2.3}$$

Function $\Psi(\gamma)$ defined by (2.1) and (2.3) lives on the universal covering X of $\mathbb{C}P^1 \setminus \{\gamma_1, \dots, \gamma_N\}$. The asymptotical expansion of $\Psi(\gamma)$ near singularities γ_j is given by

$$\Psi(\gamma) = Q_j(I + O(\gamma - \gamma_j))(\gamma - \gamma_j)^{T_j} C_j \tag{2.4}$$

where $Q_j,\ C_j\ \in SL(2, \mathbb{C})$ and T_j is traceless diagonal matrix. Matrices

$$M_j = C_j^{-1} e^{2\pi i T_j} C_j, \qquad j = 1, \dots, N \tag{2.5}$$

are called monodromy matrices.

The assumtion of independence of all matrices M_j of the parameters γ_j:

$$\frac{\partial M_j}{\partial \gamma_k} = 0 \tag{2.6}$$

is called the isomonodromy condition; it implies the following dependence of $\Psi(\gamma)$ on γ_j, which can be deduced from (2.4):

$$\frac{\partial \Psi}{\partial \gamma_j} = -\frac{A_j}{\gamma - \gamma_j}\Psi \tag{2.7}$$

The compatibility condition of (2.1) and (2.7) is equivalent to the Schlesinger system [28] for the residues A_j:

$$\frac{\partial A_j}{\partial \gamma_i} = \frac{[A_i, A_j]}{\gamma_i - \gamma_j}, \quad i \neq j \;, \qquad \frac{\partial A_i}{\partial \gamma_i} = -\sum_{j\neq i} \frac{[A_i, A_j]}{\gamma_i - \gamma_j}. \tag{2.8}$$

The Schlesinger system admits the "multi-time" Hamiltonian formulation ([32]) with respect to the following Poisson structure:

$$\{A_j^a, A_k^b\} = f_c^{ab} A_k^c \delta_{jk} \tag{2.9}$$

where f_c^{ab} are structure constants of $sl(2)$. Evolution with respect to "times" γ_j is described by the Hamiltonians

$$H_j = \sum_{k\neq j} \frac{\mathrm{tr} A_j A_k}{\gamma_j - \gamma_k} \tag{2.10}$$

The function $\tau(\{\gamma_j\})$, generating Hamiltonians H_j according to equations

$$\frac{\partial}{\partial \gamma_j} \ln \tau = H_j \tag{2.11}$$

is called the τ-function of Schlesinger system. Compatibility of equations (2.11) follows from the Poisson commutativity of all Hamiltonians H_j.

2.2 Stationary axisymmetric Einstein equations from Schlesinger system

The Einstein equations for the line element

$$ds^2 = f^{-1}[e^{2k}(dz^2 + d\rho^2) + \rho^2 d\varphi^2] - f(dt + F d\varphi)^2 \tag{2.12}$$

where all metric coefficients f, k, F are assumed to depend only on ρ and z, reduce to the Ernst equation

$$(\mathcal{E} + \bar{\mathcal{E}})(\mathcal{E}_{zz} + \frac{1}{\rho}\mathcal{E}_\rho + \mathcal{E}_{\rho\rho}) = 2(\mathcal{E}_z^2 + \mathcal{E}_\rho^2). \tag{2.13}$$

The metric coefficients may be restored from the complex-valued Ernst potential $\mathcal{E}(z, \rho)$ according to the following equations:

$$f = \Re\mathcal{E} \qquad F_\xi = 2\rho \frac{(\mathcal{E} - \bar{\mathcal{E}})_\xi}{(\mathcal{E} + \bar{\mathcal{E}})^2} \qquad k_\xi = 2i\rho \frac{\mathcal{E}_\xi \bar{\mathcal{E}}_\xi}{(\mathcal{E} + \bar{\mathcal{E}})^2} \tag{2.14}$$

where $\xi = z + i\rho$.

Equation (2.13) is the compatibility condition of the following linear system:

$$\Psi_\xi = \frac{G_\xi G^{-1}}{1-\gamma}\Psi \qquad \Psi_{\bar\xi} = \frac{G_{\bar\xi} G^{-1}}{1+\gamma}\Psi \tag{2.15}$$

where

$$\gamma = \frac{2}{\xi - \bar\xi}\left\{\lambda - \frac{\xi+\bar\xi}{2} + \sqrt{(\lambda-\xi)(\lambda-\bar\xi)}\right\}, \tag{2.16}$$

$\lambda \in \mathbb{C}$ is a spectral parameter and

$$G = \frac{1}{\mathcal{E}+\bar{\mathcal{E}}}\begin{pmatrix} 2 & i(\mathcal{E}-\bar{\mathcal{E}}) \\ i(\mathcal{E}-\bar{\mathcal{E}}) & 2\mathcal{E}\bar{\mathcal{E}} \end{pmatrix} \tag{2.17}$$

In terms of the matrix G the Ernst equation may be equivalently rewritten as follows:

$$(\rho G_\rho G^{-1})_\rho + (\rho G_z G^{-1})_z = 0 \tag{2.18}$$

Relationship between solutions of Schlesinger system and Ernst equation was revealed in [30]:

Theorem 2.1 *Let $\{A_j\}$ be some solution of the Schlesinger system (2.8) and $\Psi(\gamma)$ be related solution of equation (2.1) satisfying the following conditions:*

$$\Psi^t(\frac{1}{\gamma})\Psi(0)^{-1}\Psi(\gamma) = I\ , \tag{2.19}$$

$$\Psi(-\bar\gamma) = \overline{\Psi(\gamma)}\ . \tag{2.20}$$

Let in addition $\gamma_j = \gamma(\lambda_j, \xi, \bar\xi)$, $\lambda_j \in \mathbb{C}$ for all j, with function $\gamma(\lambda,\xi,\bar\xi)$ given by (2.16). Then

$$G(\xi,\bar\xi) \equiv \Psi(\gamma = 0, \xi, \bar\xi) \tag{2.21}$$

is a solution of Ernst equation (2.18). Function $\Psi(\gamma, \{\gamma_j\})$ solves the associated linear system (2.15). Metric coefficient e^{2k} of the line element (2.12) is related to the τ-function of the Schlesinger system as follows:

$$e^{2k} = C\prod_{j=1}^{N}\left\{\frac{\partial\gamma_j}{\partial\lambda_j}\right\}^{\mathrm{tr}A_j^2/2}\tau\ , \tag{2.22}$$

where C is a constant of integration.

3 Solutions of the Schlesinger system in terms of theta-functions

Let $N = 2\gamma + 2$. Define hyperelliptic curve $\mathcal{L}$ of genus γ by the equation

$$w^2 = \prod_{j=1}^{2g+2}(\gamma - \gamma_j) \tag{3.1}$$

with basic cycles (a_j, b_j) chosen according to figure 1.

The basic holomorphic 1-forms on $\mathcal{L}$ are given by

$$\frac{\gamma^{k-1}d\gamma}{w}, \qquad k=1,\dots,g. \tag{3.2}$$

Let us define $g\times g$ matrices of a- and b-periods of these 1-forms by

$$\mathcal{A}_{kj}=\oint_{a_j}\frac{\gamma^{k-1}d\gamma}{w}, \qquad \mathcal{B}_{kj}=\oint_{b_j}\frac{\gamma^{k-1}d\gamma}{w}. \tag{3.3}$$

Then the holomorphic 1-forms

$$dU_k=\frac{1}{w}\sum_{j=1}^{g}(\mathcal{A}^{-1})_{kj}\gamma^{j-1}d\gamma \tag{3.4}$$

satisfy the normalization conditions $\oint_{a_j} dU_k=\delta_{jk}$.

The matrices $\mathcal{A}$ and $\mathcal{B}$ define the symmetric $g\times g$ matrix of b-periods of the curve $\mathcal{L}$:

$$\mathbf{B}=\mathcal{A}^{-1}\mathcal{B}\ .$$

Let us now introduce the theta function with characteristic $[{}^{\mathbf{p}}_{\mathbf{q}}]$ ($\mathbf{p}\in\mathbb{C}^g$, $\mathbf{q}\in\mathbb{C}^g$) by the following series,

$$\Theta[{}^{\mathbf{p}}_{\mathbf{q}}](\mathbf{z}|\mathbf{B})=\sum_{\mathbf{m}\in\mathbb{Z}^g}\exp\{\pi i\langle\mathbf{B}(\mathbf{m}+\mathbf{p}),\mathbf{m}+\mathbf{p}\rangle+2\pi i\langle\mathbf{z}+\mathbf{q},\mathbf{m}+\mathbf{p}\rangle\}, \tag{3.5}$$

for any $\mathbf{z}\in\mathbb{C}^g$. It possesses the following periodicity properties:

$$\Theta[{}^{\mathbf{p}}_{\mathbf{q}}](\mathbf{z}+\mathbf{e}_j)=e^{2\pi i p_j}\Theta[{}^{\mathbf{p}}_{\mathbf{q}}](\mathbf{z}), \tag{3.6}$$

$$\Theta\left[{}^{\mathbf{p}}_{\mathbf{q}}\right](\mathbf{z}+\mathbf{B}\mathbf{e}_j)=e^{-2\pi i q_j}e^{-\pi i\mathbf{B}_{jj}-2\pi i\mathbf{z}_j}\Theta[{}^{\mathbf{p}}_{\mathbf{q}}](\mathbf{z}), \tag{3.7}$$

where

$$\mathbf{e}_j\equiv(0,\dots,1,\dots,0)^t \tag{3.8}$$

(1 stands in the jth place).

The theta-function with characteristics is related as follows to the theta-function without characteristics:

$$\Theta[{}^{\mathbf{p}}_{\mathbf{q}}](\mathbf{z})=\Theta(\mathbf{z}+\mathbf{B}\mathbf{p}+\mathbf{q})e^{\pi i\langle\mathbf{B}\mathbf{p},\mathbf{p}\rangle+2\pi i\langle\mathbf{p},\mathbf{z}+\mathbf{q}\rangle} \tag{3.9}$$

Cut curve $\mathcal{L}$ along all basic cycles to get the fundamental polygon $\hat{\mathcal{L}}$. For any meromorphic 1-form dW on $\mathcal{L}$ we shall use the notation

$$W|_Q^P\equiv\int_Q^P dW$$

where the integration contour lies inside of $\hat{\mathcal{L}}$ (if dW is meromorphic, the value of this integral might also depend on the choice of integration contour inside of $\hat{\mathcal{L}}$). By $U|_Q^P$ we shall denote vector with components $U_j|_Q^P$. The vector of Riemann constants corresponding to our choice of the initial point of this map reads as follows [31]:

$$K=\frac{1}{2}\mathbf{B}(\mathbf{e}_1+\ldots+\mathbf{e}_g)+\frac{1}{2}(\mathbf{e}_1+2\mathbf{e}_2\ldots+g\mathbf{e}_g). \tag{3.10}$$

The characteristic with components $\mathbf{p} \in \mathbb{C}^g/2\mathbb{C}^g$, $\mathbf{q} \in \mathbb{C}^g/2\mathbb{C}^g$ is called half-integer characteristic: the half-integer characteristics are in one-to-one correspondence with the half-periods $\mathbf{B}\mathbf{p}+\mathbf{q}$. If the scalar product $4\langle \mathbf{p}, \mathbf{q}\rangle$ is odd, then the related theta function is odd with respect to its argument $\mathbf{z}$ and the characteristic $[^{\mathbf{p}}_{\mathbf{q}}]$ is called odd, and if this scalar product is even, then the theta function $\Theta[^{\mathbf{p}}_{\mathbf{q}}](\mathbf{z})$ is even with respect to $\mathbf{z}$ and the characteristic $[^{\mathbf{p}}_{\mathbf{q}}]$ is called even.

The odd characteristics which will be of importance for us in the sequel correspond to any given subset $S = \{\gamma_{i_1}, \ldots, \gamma_{i_{g-1}}\}$ of $g-1$ arbitrary non-coinciding branch points. The odd half-period associated to the subset S is given by

$$\mathbf{B}\mathbf{p}^S + \mathbf{q}^S = \sum_{j=1}^{g-1} U\Big|_{\gamma_1}^{\gamma_{i_j}} - K. \tag{3.11}$$

where $dU = (dU_1, \ldots, dU_g)^t$. Denote by $\Omega_\gamma \subset \mathbb{C}$ the neighbourhood of the infinite point $\gamma = \infty$, such that Ω_γ does not overlap with projections of all basic cycles on γ-plane. Let the 2×2 matrix-valued function $\Phi(\gamma)$ be defined in the domain Ω_γ of the first sheet of $\mathcal{L}$ by the following formula,

$$\Phi(\gamma \in \Omega_\gamma) = \begin{pmatrix} \varphi(\gamma) & \varphi(\gamma^*) \\ \psi(\gamma) & \psi(\gamma^*) \end{pmatrix}, \tag{3.12}$$

where functions φ and ψ are defined in the fundamental polygon $\hat{\mathcal{L}}$ by the formulas:

$$\varphi(\gamma) = \Theta\left[^{\mathbf{p}}_{\mathbf{q}}\right]\left(U|_{\gamma_1}^{\gamma} + U|_{\gamma_1}^{\gamma_\varphi}\Big|\mathbf{B}\right)\Theta[S]\left(U|_{\gamma_\varphi}^{\gamma}\Big|\mathbf{B}\right), \tag{3.13}$$

$$\psi(\gamma) = \Theta\left[^{\mathbf{p}}_{\mathbf{q}}\right]\left(U|_{\gamma_1}^{\gamma} + U|_{\gamma_1}^{\gamma_\psi}\Big|\mathbf{B}\right)\Theta[S]\left(U|_{\gamma_\psi}^{\gamma}\Big|\mathbf{B}\right), \tag{3.14}$$

with two arbitrary (possibly $\{\gamma_j\}$-dependent) points γ_φ, $\gamma_\psi \in \mathcal{L}$ and arbitrary constant characteristic $\left[^{\mathbf{p}}_{\mathbf{q}}\right]$; $*$ is the involution on $\mathcal{L}$ interchanging the sheets;

$$\Theta[S](\mathbf{z}) \equiv \Theta\begin{bmatrix}\mathbf{p}^S\\ \mathbf{q}^S\end{bmatrix}(\mathbf{z})$$

where odd theta characteristic $\begin{bmatrix}\mathbf{p}^S\\ \mathbf{q}^S\end{bmatrix}$ corresponds to an arbitrary subset S of $g-1$ branch points via Eq. (3.11).

Since domain Ω_γ does not overlap with projections of all basic cycles of $\mathcal{L}$ on γ-plane, domain $\mathcal{L}_\gamma^*$ does not overlap with the boundary of $\hat{\mathcal{L}}$, and functions $\varphi(\gamma^*)$ and $\psi(\gamma^*)$ in (3.12) are uniquely defined by (3.13), (3.14) for $\gamma \in \Omega_\gamma$.

Now choose some sheet of the universal covering X, define new function $\Psi(\gamma)$ in subset Ω_γ of this sheet by the formula

$$\Psi(\gamma \in \Omega_\gamma) = \sqrt{\frac{\det\Phi(\infty^1)}{\det\Phi(\gamma)}}\,\Phi^{-1}(\infty^1)\Phi(\gamma) \tag{3.15}$$

and extend on the rest of X by analytical continuation.

Function $\Psi(\gamma)$ (3.15) transforms as follows with respect to the tracing around basic cycles of $\mathcal{L}$ (by T_{a_j} and T_{b_j} we denote corresponding transport operators):

$$T_{a_j}[\Psi(\gamma)] = \Psi(\gamma)M_{a_j}\ ; \qquad T_{b_j}[\Psi(\gamma)] = \Psi(\gamma)M_{b_j}\ , \qquad j = 1, \ldots, 2g_0 - 1\ ,$$

where

$$M_{a_j} = \begin{pmatrix} e^{2\pi i p_j} & 0 \\ 0 & e^{-2\pi i p_j} \end{pmatrix} , \qquad M_{b_j} = \begin{pmatrix} e^{-2\pi i q_j} & 0 \\ 0 & e^{2\pi i q_j} \end{pmatrix} . \tag{3.16}$$

The following statement was proved in the paper [29]:

Theorem 3.1 *Let $\mathbf{p}, \mathbf{q} \in \mathbb{C}^g$ be an arbitrary set of $2g$ constants such that $\begin{bmatrix}\mathbf{p}\\ \mathbf{q}\end{bmatrix}$ is not half-integer characteristic. Then:*

1. *Function $\Psi(Q \in X)$ defined by (3.15) is independent of γ_φ and γ_ψ and satisfies the linear system (2.1) with*

$$A_j \equiv \mathrm{res}|_{\gamma=\gamma_j} \left\{ \Psi_\gamma \Psi^{-1} \right\} . \tag{3.17}$$

   which in turn solve the Schlesinger system (2.8).

2. *Monodromies (2.5) of $\Psi(\gamma)$ around points γ_j are given by*

$$M_j = \begin{pmatrix} 0 & -m_j \\ m_j^{-1} & 0 \end{pmatrix} , \tag{3.18}$$

   where constants m_j may be expressed in terms of $\mathbf{p}$ and $\mathbf{q}$ (see [29]).

3. *The τ-function, corresponding to solution (3.17) of the Schlesinger system, has the following form:*

$$\tau(\{\gamma_j\}) = [\det \mathcal{A}]^{-\frac{1}{2}} \prod_{j<k} (\gamma_j - \gamma_k)^{-\frac{1}{8}} \Theta \begin{bmatrix}\mathbf{p}\\ \mathbf{q}\end{bmatrix} (0|\mathbf{B}) \tag{3.19}$$

4 Solutions of the Ernst equation in terms of theta-functions. Formulas for the metric coefficients

According to the relationship between Schlesinger system and Ernst equation given by the theorem 2.1, we can derive solutions of Ernst equation in terms of theta-functions from construction of the theorem 3.1. The necessary additional work to do is to choose the parameters of the construction (i.e. the constants λ_j and vectors $\mathbf{p}$, $\mathbf{q}$) to provide the constraints (2.19) and (2.20). To get these constraints fulfilled we have to assume that the curve $\mathcal{L}$ is invariant under the holomorphic involution σ acting on every sheet of $\mathcal{L}$ as

$$\sigma \; : \; \gamma \to \frac{1}{\gamma} , \tag{4.1}$$

and anti-holomorphic involution μ acting on every sheet of $\mathcal{L}$ as

$$\mu \; : \; \gamma \to -\bar{\gamma} . \tag{4.2}$$

Constraints (2.19) and (2.20) turn out to be compatible with each other only if the genus g is odd:

$$g = 2g_0 - 1 . \tag{4.3}$$

We shall enumerate the branch points γ_j, $j = 1, \dots, 4g_0$ in such an order that

$$\gamma_j = \gamma_{j+2g_0}^{-1} , \qquad j = 1, \dots, 2g_0 \tag{4.4}$$

and for some $k \leq g_0$

$$\gamma_j \in i\mathbb{R} , \quad 1 \leq j \leq 2k ; \qquad \gamma_{2j+1} + \bar{\gamma}_{2j+2} = 0 , \quad 2k+1 \leq j \leq 2g_0 - 1.$$

We shall now distinquish two cases:

1. Curve $\mathcal{L}$ is non-separable [4] with respect to the action of anti-involution μ, i.e. $k \geq 1$. Then the basic cycles (a_j, b_j) , $j = 1, \ldots, 2g_0 - 1$ on $\mathcal{L}$ may be chosen as shown in figure 2a.

2. Curve $\mathcal{L}$ is separable with respect to the action of anti-involution μ, i.e. $k = 0$ and, therefore, none of the points γ_j lie on the imaginary axis. In this case we choose the basic cycles on $\mathcal{L}$ as shown in figure 2b.

In both cases the basic cycles transform in the following way under the action of the involution σ:

$$\sigma(a_1) = -a_1 \qquad \sigma(b_1) = -b_1 \tag{4.5}$$

$$\sigma(a_j) = a_{j+g_0-1} \qquad \sigma(b_j) = b_{j+g_0-1} \ , \qquad 2 \leq j \leq g_0 \tag{4.6}$$

Constraint (2.19) leads to the following equations for M_{a_j} and M_{b_j} (3.16):

$$M_{a_j}^t M_{a_{j+g_0-1}} = I \ ; \qquad M_{b_j}^t M_{b_{j+g_0-1}} = I \ , \qquad 2 \leq j \leq g_0 \ .$$

(Equations $M_{a_1}^t = M_{a_1}$ and $M_{b_1}^t = M_{b_1}$, which arise from the calculation of monodromies of (2.19) along the basic cycles a_1 and b_1, are identically fulfilled)

In turn, we get for components of the vectors $\mathbf{p}$ and $\mathbf{q}$ the following equations

$$p_j + p_{j+g_0-1} = 0 \ ; \qquad q_j + q_{j+g_0-1} = 0 \ , \qquad 2 \leq j \leq g_0 \tag{4.7}$$

It remains to derive constraints on $\mathbf{p}$ and $\mathbf{q}$ imposed by the "reality conditions" (2.20). We shall consider two cases separately.

1. *Non-separable case ($k \geq 1$).* The basic cycles of $\mathcal{L}$ shown in figure 2a behave as follows with respect to the action of the anti-involution μ:

$$\mu(a_j) = -a_j \ , \quad \forall j \ ; \qquad \mu(b_1) = b_1 + 2a_1$$

$$\mu(b_j) = b_j \ , \quad 2 \leq j \leq k \ ; \qquad \mu(b_j) = b_j - a_j \ , \quad k+1 \leq j \leq g_0$$

(since we assumed already the invariance of $\mathcal{L}$ under the involution σ, it is sufficient to determine the action of μ only on the first g_0 cycles b_j). Now from (2.20) we get the following equations for monodromies:

$$\overline{M}_{a_j} = M_{a_j}^{-1} \ , \quad \forall j \ ; \qquad \overline{M}_{b_1} = M_{b_1} M_{a_1}^2 \ ;$$

$$\overline{M}_{b_j} = M_{b_j} \ , \quad 2 \leq j \leq k \ ; \qquad \overline{M}_{b_j} = M_{b_j} M_{a_j}^{-1} \ , \quad k+1 \leq j \leq g_0 \ ,$$

which is equivalent to the following conditions imposed on $\mathbf{p}$ and $\mathbf{q}$:

$$p_j \in \mathbb{R} \ , \quad \forall j; \qquad \Re q_1 = p_1 \ ;$$

$$\Re q_j = 0 \ , \quad 2 \leq j \leq k \ ; \qquad \Re q_j = -\frac{1}{2} p_j \ , \quad k+1 \leq j \leq g_0 \ . \tag{4.8}$$

[4] Curve admitting anti-holomorphic involution is called separable, if it is divided into two pieces by the ovals invariant with respect to the anti-involution, and non-separable otherwise.

2. *Separable case ($k = 0$).* In this case we shall choose the basic cycles according to figure 2b. They transforms under the action of μ in the following way:

$$\mu(a_j) = -a_j \ , \quad \forall j \ ; \qquad \mu(b_1) = b_1 + \sum_{l=2}^{g_0} (a_{l+g_0-1} - a_l) \ ;$$

$$\mu(b_j) = b_j - a_j + \sum_{l=1}^{g_0} a_l \ , \quad 2 \le j \le g_0 \ ,$$

which leads to the following conditions on $\mathbf{p}$ and $\mathbf{q}$:

$$p_j \in \mathbb{R} \ , \quad \forall j \ ; \qquad \Re q_1 = -\frac{1}{2}\sum_{l=2}^{g_0} p_l \ ; \qquad \Re q_j = -\frac{1}{2}\sum_{l=1}^{g_0} p_l - \frac{p_j}{2} \ . \tag{4.9}$$

Now, taking into account the theorems 2.1 and 3.1, we get the following

Theorem 4.1 *Let the genus of curve $\mathcal{L}$ be odd: $g = 2g_0 - 1$; and the basic cycles be chosen according to figure 2a if $\mathcal{L}$ is of non-separable and figure 2b if $\mathcal{L}$ is of separable type. Let $\mathbf{p}, \mathbf{q} \in \mathbb{C}^{2g_0-1}$ be arbitrary constant vectors satisfying conditions (4.7). Let in addition $\mathbf{p}, \mathbf{q}$ satisfy conditions (4.8) if $\mathcal{L}$ is non-separable and (4.9) if $\mathcal{L}$ is separable. Define function $\Psi(\gamma)$ by the expressions (3.12), (3.13),(3.14) and (3.15). Then the function*

$$G(x, \rho) \equiv \Psi(x, \rho, \gamma = 0) \tag{4.10}$$

satisfies the Ernst equation (2.18) and may be represented in the form (2.17).

The expression for the Ernst potential $\mathcal{E}$ may be obtained from (4.10) in terms of theta-functions associated to curve $\mathcal{L}$. However, it may be essentially simplified if we make use of invariance of curve $\mathcal{L}$ under involution σ and use the spectral parameter λ from (2.16) instead of γ. Namely, curve $\mathcal{L}$ may be represented as fourfold covering of λ-plane; its Hurvitz diagram is shown in figure 3. In this realization involution σ interchanges the sheets $1 \leftrightarrow 3$ and $2 \leftrightarrow 4$; involution $*$ interchanges the sheets $1 \leftrightarrow 2$ and $3 \leftrightarrow 4$.

Now introduce new hyperelliptic curve $\mathcal{L}_0$ of genus g_0 defined by equation

$$\nu^2 = (\lambda - \xi)(\lambda - \bar{\xi}) \prod_{j=1}^{2g_0} (\lambda - \lambda_j) \tag{4.11}$$

Curve $\mathcal{L}_0$ together with canonical basis of cycles (a_j^0, b_j^0) is shown in figure 4. Introduce the dual basis $dV = (dV_1, \dots, dV_{g_0})^t$ of holomorphic 1-forms on $\mathcal{L}_0$ by

$$dV_j = \frac{1}{\nu} \sum_{k=1}^{g_0} (\mathcal{A}_0^{-1})_{jk} \lambda^{k-1} d\lambda \ , \quad j = 1, \dots, g_0 \ , \tag{4.12}$$

where

$$(\mathcal{A}_0)_{kj} \equiv \oint_{a_j^0} \frac{\lambda^{k-1} d\lambda}{\nu} \ , \quad j, k = 1, \dots, g_0 \ , \tag{4.13}$$

and corresponding matrix of b-periods $\mathbf{B}_0$.

Curve $\mathcal{L}$ is twofold non-ramified covering of $\mathcal{L}_0$: $\Pi : \mathcal{L} \to \mathcal{L}_0$, such that the points of $\mathcal{L}$ related by the involution σ project onto the same point of $\mathcal{L}_0$, namely, the λ-sheets 1 and 3 of $\mathcal{L}$ are projected onto the 1st sheet of $\mathcal{L}_0$, and sheets 2 and 4 of $\mathcal{L}$ are projected onto the 2nd sheet of $\mathcal{L}_0$, preserving projections of corresponding points on λ-plane. Anti-involution μ inherited from $\mathcal{L}$ acts on every sheet of $\mathcal{L}_0$ as $\lambda \to \bar{\lambda}$.

Existence of reduction (2.19) allows to alternatively express function Ψ in terms of theta-functions associated to the curve $\mathcal{L}_0$. Denote by Ω_λ the neighbourhood of the point $\lambda = \infty$ being the projection of the domain Ω_γ into λ-plane. Define function $\Phi_0(\lambda \in \Omega_\lambda)$ in the domain Ω_λ lying on the first sheet of $\mathcal{L}_0$ by the following formula:

$$\Phi_0(\lambda \in \Omega_\lambda) = \begin{pmatrix} \varphi_0(\lambda) & \varphi_0(\lambda^*) \\ \psi_0(\lambda) & \psi_0(\lambda^*) \end{pmatrix}, \tag{4.14}$$

where involution $*$ inherited on $\mathcal{L}_0$ from $\mathcal{L}$ interchanges the λ-sheets of $\mathcal{L}_0$;

$$\varphi_0(\lambda) = \Theta \begin{bmatrix} \mathbf{r} \\ \mathbf{s} \end{bmatrix} \left(V|_\xi^\lambda \middle| \mathbf{B}_0 \right) , \qquad \psi_0(\lambda) = \Theta \begin{bmatrix} \mathbf{r} \\ \mathbf{s} \end{bmatrix} \left(V|_{\bar{\xi}}^\lambda \middle| \mathbf{B}_0 \right) , \tag{4.15}$$

$$\varphi_0(\lambda^*) = -i\Theta \begin{bmatrix} \mathbf{r} \\ \mathbf{s} \end{bmatrix} \left(-V|_\xi^\lambda \middle| \mathbf{B}_0 \right) , \qquad \psi_0(\lambda^*) = i\Theta \begin{bmatrix} \mathbf{r} \\ \mathbf{s} \end{bmatrix} \left(-V|_{\bar{\xi}}^\lambda dV \middle| \mathbf{B}_0 \right) , \tag{4.16}$$

for $\lambda \in \Omega_\lambda$, and constant vectors $\mathbf{r}$, $\mathbf{s} \in \mathbb{C}^{g_0}$ satisfy the following reality conditions:

$$\mathbf{r} \in \mathbb{R}^{g_0} \tag{4.17}$$

$$\Re \mathbf{s}_j = \sum_{l=1}^{g_0} \frac{\mathbf{r}_l}{2} , \quad 1 \le j \le k ; \qquad \Re \mathbf{s}_j = \sum_{l=1,\, l\neq j}^{g_0} \frac{\mathbf{r}_l}{2} , \quad k+1 \le j \le g_0 . \tag{4.18}$$

The basic cycles of curve $\mathcal{L}_0$ behave as follows under the action of anti-involution μ:

$$\mu(a_j^0) = -a_j^0 , \quad \forall j ; \qquad \mu(b_j^0) = b_j^0 + \sum_{l=1}^{g_0} a_l^0 , \quad j \le k ; \qquad \mu(b_j^0) = b_j^0 + \sum_{l\neq j} a_l^0 , \quad j > k$$

which implies the following relations for the matrix of b-periods of $\mathcal{L}_0$:

$$\Re(\mathbf{B}_0)_{jl} = -\frac{1}{2} , \quad j \le k ; \qquad \Re(\mathbf{B}_0)_{jl} = -\frac{1}{2} + \frac{\delta_{jl}}{2} , \quad j > k$$

Now the relations (4.17), (4.18) may be equivalently represented as

$$\Re(\mathbf{B}_0 \mathbf{r} + \mathbf{s}) = 0 ; \tag{4.19}$$

this, in particular, implies the following relation between functions φ_0 and ψ_0:

$$\psi_0(P) = \overline{\varphi_0(\bar{P})} \tag{4.20}$$

Now define function

$$\Psi_0(\lambda \in \Omega_\lambda) \equiv \sqrt{\frac{\det \Phi_0(\infty^1)}{\det \Phi_0(\lambda)}} \Phi_0^{-1}(\infty^1) \Phi_0(\lambda) , \tag{4.21}$$

and extend it to the universal covering X by analytical continuation to get function $\Psi_0(P \in X)$. The following statement takes place:

Theorem 4.2 *Let function $\Psi(P \in X)$ be defined by the formulas (3.12), (3.13), (3.14), (3.15), and function $\Psi_0(P \in X)$ be defined by formulas (4.14), (4.15), (4.21). Let the components of vectors $\mathbf{p}, \mathbf{q} \in \mathbb{C}^{2g_0-1}$ and $\mathbf{r}, \mathbf{s} \in \mathbb{C}^{g_0}$ be related as follows:*

$$p_1 = -\sum_{l=1}^{g_0} r_l \;, \quad q_1 = -2s_1 \;; \qquad p_j = r_j \;, \quad q_j = s_j - s_1 \;, \quad 2 \le j \le g_0 \tag{4.22}$$

(relations (4.7) automatically give other $g_0 - 1$ components of $\mathbf{p}$ and $\mathbf{q}$). Require in addition that $\left[\begin{smallmatrix}\mathbf{r}\\ \mathbf{s}\end{smallmatrix}\right]$ is not a half-integer characteristic.

Then functions Ψ and Ψ_0 coincide:

$$\Psi(P \in X) = \Psi_0(P \in X) \;. \tag{4.23}$$

Proof. The functions φ_0 and ψ_0 were chosen in such a way that the analytical continuation of Ψ_0 from Ω_λ on the whole universal covering X does not violate reduction restriction (2.19). Taking into account the non-triviality assumption of non-coincidence of characteristic $\left[\begin{smallmatrix}\mathbf{r}\\ \mathbf{s}\end{smallmatrix}\right]$ with any half-integer characteristic, and coincidence of the normalization conditions of Ψ and Ψ_0 at $\gamma = \infty$, it is enough to show that monodromies of Ψ and Ψ_0 along the first g_0 pairs of basic cycles of curve $\mathcal{L}$ coincide. For our choice of the basic cycles on $\mathcal{L}$ and $\mathcal{L}_0$ we get the following relations between the basic cycles of $\mathcal{L}_0$ and $\mathcal{L}$:

$$\Pi(a_1) = -(a_1^0 + \ldots + a_{g_0}^0) \;, \qquad \Pi(b_1) = -2b_1^0 \;, \qquad \Pi(a_j) = a_j^0 \;, \qquad \Pi(b_j) = b_j^0 - b_1^0 \;,$$

which imply the following expressions for monodromies of Ψ_0 around (a_j, b_j):

$$M_{a_1}^0 = \exp\{-2\pi i \sum_{j=1}^{g_0} r_j \sigma_3\} \;, \qquad M_{b_1}^0 = \exp\{4\pi i s_1 \sigma_3\} \tag{4.24}$$

$$M_{a_j}^0 = \exp\{2\pi r_j \sigma_3\} \qquad M_{b_j}^0 = \exp\{2\pi (s_1 - s_j)\sigma_3\} \;, \quad 2 \le j \le g_0 \tag{4.25}$$

coinciding with monodromies of Ψ (3.16) provided conditions (4.22) are fulfilled (reduction (2.19) provides coicidence of remaining monodromies around (a_j, b_j), $g_0 + 1 \le j \le 2g_0$). [Notice that functions φ_0 and ψ_0 transform in a different way (their monodromies differ by sign) along the basic cycles of $\mathcal{L}_0$, but their pullbacks on $\hat{\mathcal{L}}$ tranform in the same way along the basic cycles of $\mathcal{L}$.]

Now we are in position to formulate the following statement:

Theorem 4.3 *Let $\mathbf{r}, \mathbf{s} \in \mathbb{C}^{g_0}$ be arbitrary constant vectors satisfying reality conditions (4.17), (4.18). Then the following function:*

$$\mathcal{E}(\xi, \bar{\xi}) = \frac{\Theta\left[\begin{smallmatrix}\mathbf{r}\\ \mathbf{s}\end{smallmatrix}\right]\left(V|_{\xi}^{\infty^1} \Big| \mathbf{B}_0\right)}{\Theta\left[\begin{smallmatrix}\mathbf{r}\\ \mathbf{s}\end{smallmatrix}\right]\left(V|_{\xi}^{\infty^2} \Big| \mathbf{B}_0\right)} \;, \tag{4.26}$$

solves the Ernst equation (2.13). Function Ψ defined equivalently by (3.15) or (4.21) satisfies the linear system (2.15) with matrix $G(\xi, \bar{\xi})$ given by (2.17), and

$$\Psi(\gamma = \infty) = G \tag{4.27}$$

Proof. The non-trivial part is to check (4.26). One can represent matrix $\Psi(\lambda = \infty^1)$, given by (4.21), in the form (2.17) with

$$\mathcal{E} = -i\frac{\varphi_0(\infty^1)}{\varphi_0(\infty^2)}$$

which leads to (4.26) after substitution of (4.15).

Now it arises the non-trivial problem of integration of equations (2.14) for the metric coefficients F and k. The following statement shows how to find the metric coefficient $F(\xi,\bar{\xi})$ corresponding to the Ernst potential (4.26):

Theorem 4.4 *The metric coefficient $F(\xi,\bar{\xi})$ (2.14) corresponding to the Ernst potential (4.26) is given by*

$$F = \frac{2}{\Re\mathcal{E}}\Im\Big\{\sum_{j=1}^{g_0}(\mathcal{A}_0^{-1})_{g_0 j}\frac{\partial}{\partial z_j}\ln\Theta\left[\begin{smallmatrix}\mathbf{r}\\ \mathbf{s}\end{smallmatrix}\right]\left(V|_\xi^{\infty^2}\Big|\mathbf{B}_0\right)\Big\} \tag{4.28}$$

up to an arbitrary additive constant, where $\mathcal{A}_0$ is the matrix of a-periods of holomorphic 1-forms (4.13); $\frac{\partial\Theta}{\partial z_j}$ denotes derivative of theta-function with respect to its jth argument.

Proof. Consider the following simple identity

$$(\Psi^{-1}\Psi_{\lambda^{-1}})_\xi \equiv \Psi^{-1}\left(\Psi_\xi\Psi^{-1}\right)_{1/\lambda}\Psi \tag{4.29}$$

Evaluation of (12) matrix element of the right hand side of (4.29) at $\gamma = \infty$ using the linear system (2.15), equation for F (2.14) and normalization condition (2.3) gives nothing but $\frac{1}{2}F_\xi$. Therefore, we can integrate equations for F to get

$$F = 2\Big(\Psi^{-1}\Psi_{\lambda^{-1}}\Big)_{12}(\gamma = \infty) \tag{4.30}$$

Substitution of (4.15), (4.16) and (4.21) into (4.30) leads to the following expression:

$$F = -\frac{2}{\Re\mathcal{E}}\Im\Big\{\frac{dW(P)}{d\frac{1}{\lambda}}(\lambda = \infty^2)\Big\}$$

where the 1-form $dW_0(P \in \mathcal{L}_0)$ is given by

$$dW_0(P) = \sum_{j=1}^{g_0} dV_j(P)\frac{\partial}{\partial z_j}\ln\Theta\left[\begin{smallmatrix}\mathbf{r}\\ \mathbf{s}\end{smallmatrix}\right]\left(V|_\xi^P\Big|\mathbf{B}_0\right)$$

Using formula (4.12) for basic differentials dV_j , we come to (4.28).

The next theorem shows how to integrate equation (2.14) for remaining metric coefficient k:

Theorem 4.5 *The metric coefficient $e^{2k(\xi,\bar{\xi})}$, solving equation (2.14) with Ernst potential defined by (4.26), is given by the following expression:*

$$e^{2k} = \frac{\Theta\left[\begin{smallmatrix}\mathbf{p}\\ \mathbf{q}\end{smallmatrix}\right](0|\mathbf{B})}{\sqrt{\det\mathcal{A}_0}}\prod_{j=1}^{2g_0}|\lambda_j - \xi|^{-1/4} \tag{4.31}$$

up to multiplication with an arbitrary constant, where $\mathbf{B}$ *is the matrix of* b*-periods of curve* $\mathcal{L}$ *(3.1); matrix* $\mathcal{A}_0$ *of* a*-periods of holomorphic differentials on the curve* $\mathcal{L}_0$ *is defined by (4.13); the constant vectors* $\mathbf{p}, \mathbf{q} \in \mathbb{C}^{2g_0-1}$ *are expressed via (4.22) in terms of vectors* $\mathbf{r}, \mathbf{s} \in \mathbb{C}^{g_0}$*.*

Proof. One should exploit the coincidence of functions Ψ_0 and Ψ, and substitute the formula for τ-function of the Schlesinger system (3.19) into the relation (2.22) between the τ-function and the coefficient e^{2k} (in the present case $\mathrm{tr} A_j^2 = 1/8$). In addition we have to use the following relation between determinants of matrices of a-periods of holomorphic 1-forms of curves $\mathcal{L}$ and $\mathcal{L}_0$:

$$\det \mathcal{A} = const\ \rho^{g_0^2} \det \mathcal{A}_0 \tag{4.32}$$

where the constant is independent of $(\xi, \bar{\xi})$. Proof of (4.32) may be obtained by elementary manipulations using decomposition of holomorphic 1-forms on $\mathcal{L}$ into combination of holomorphic 1-forms on $\mathcal{L}_0$ and holomorphic 1-forms on $(\xi, \bar{\xi})$-independent hyperelliptic curve of genus $g_0 - 1$ defined by the equation

$$\delta^2 = \prod_{j=1}^{2g_0} (\lambda - \lambda_j)\ . \tag{4.33}$$

To complete the proof of (4.31) it remains to make use of the folowing relations:

$$\gamma^{-1}\gamma_\lambda = \{(\lambda - \xi)(\lambda - \bar{\xi})\}^{-1/2}\ ; \qquad \prod_{j<k} (\gamma_j - \gamma_k) = const\ \rho^{-4g_0^2} \prod_{j=1}^{2g_0} \{(\lambda_j - \xi)(\lambda_j - \bar{\xi})\}^{1/2}\ .$$

Remark 4.1 Using the standard expression for Vandermonde determinant, we get the following formula for $\det \mathcal{A}_0$:

$$\det \mathcal{A}_0 = \oint_{a_1} \dots \oint_{a_{g_0}} \frac{\prod_{k<j} (\mu_j - \mu_k)}{\prod_{j=1}^{g_0} \nu(\mu_j)} d\mu_1 \dots d\mu_{g_0} \tag{4.34}$$

Remark 4.2 The theta-function $\Theta \left[\begin{smallmatrix} \mathbf{p} \\ \mathbf{q} \end{smallmatrix} \right] (0|\mathbf{B})$ from (4.31) may be decomposed into combination of theta-functions associated to double matrices of b-periods $2\mathbf{B}_0$ and $2\mathbf{B}_1$ of curves (4.11) and (4.33) (see [31]).

Remark 4.3 Half-integer $\mathbf{r}$ and $\mathbf{s}$.

Remark 4.4 Degeneration $\lambda_{2j+1},\ \lambda_{2j+1} \to \kappa_j \in \mathbb{R}$ of the spectral curve $\mathcal{L}_0$ gives after appropriate choice of $[\mathbf{r}, \mathbf{s}]$ the family of multi-Kerr-NUT solutions. In particular, to get Kerr-NUT soultion itself one has to take $g_0 = 2$ and

$$r_j = \frac{n_j}{2}\ ; \quad s_j = \frac{n_j}{4} + i\alpha_j, \quad \alpha_j \in \mathbb{R}\ , \quad n_j = \pm 1\ , \quad j = 1, 2$$

Then (4.26) turns into [20]:

$$\mathcal{E} = \frac{1-\Gamma}{1+\Gamma}\,, \qquad \Gamma^{-1} = \frac{i(a_1-a_2)}{1-a_1a_2+i(a_1+a_2)}X + \frac{1+a_1a_2}{1-a_1a_2+i(a_1+a_2)}Y$$

where $a_j = n_j e^{-2\pi i\alpha_j}$, $j=1,2$, and

$$X = \frac{1}{\kappa_1-\kappa_2}\left\{\sqrt{(\kappa_1-\xi)(\kappa_1-\bar\xi)} + \sqrt{(\kappa_2-\xi)(\kappa_2-\bar\xi)}\right\}$$

$$Y = \frac{1}{\kappa_1-\kappa_2}\left\{\sqrt{(\kappa_1-\xi)(\kappa_1-\bar\xi)} - \sqrt{(\kappa_2-\xi)(\kappa_2-\bar\xi)}\right\}$$

are prolate ellipsoidal coordinates.

Remark 4.5 Elliptic case: $r=0$ - "toron" solution; $r \neq 0$-non-asymptotically flat.

5 Known form of algebro-geometric solutions of Ernst equation

The original algebro-geometric solution of Ernst equation found in [20] looks as follows in notations introduced in [22, 25]:

$$\mathcal{E} = \frac{\Theta\left(V|_{\xi}^{\infty^1} + B_W \middle| \mathbf{B}_0\right)}{\Theta\left(V|_{\xi}^{\infty^2} + B_W \middle| \mathbf{B}_0\right)} \exp\left\{W|_{\infty^2}^{\infty^1}\right\} \tag{5.1}$$

where $dW(P)$ is an arbitrary 1-form on $\mathcal{L}_0$ satisfying the following conditions:

1. dW is an arbitrary finite, infinite or continuous linear combination, with $(\xi,\bar\xi)$-independent coefficients, of normalized (all a-periods vanish) Abelian differentials on $\mathcal{L}_0$ of the 2nd and 3rd kind with $(\xi,\bar\xi)$-independent poles and singular parts.

2. Reality condition:

$$dW(\bar P) = \overline{dW(P)}\,, \qquad 2\pi i B_W \in \mathbb{R}^{g_0}\,. \tag{5.2}$$

Vector $2\pi i B_W$ is the vector of b-periods of differential dW:

$$2\pi i (B_W)_j = \oint_{b_j} dW(P) \tag{5.3}$$

5.1 Role of 1-form dW

5.1.1 Static solutions ($g_0 = 0$)

To clarify the role of differential dW we shall consider first the simplest case $g=0$, when (5.1) becomes real:

$$\ln\mathcal{E} = W|_{\infty^2}^{\infty^1} \tag{5.4}$$

which is real as a corollary of (5.2), and the Ernst equation linearises to the Euler-Darboux equation:

$$(\partial_\rho^2 + \frac{1}{\rho}\partial_\rho + \partial_z^2)\ln\mathcal{E} = 0 \tag{5.5}$$

The natural question which was asked in [22] is whether it is possible to represent general solution of (5.5) invariant with respect to the change of sign of ρ in the form (5.4). The positive answer was given in [25].

Namely, if we fix some domain D in (x,ρ)-plane, symmetric with respect to involution $\rho \to -\rho$, (which may contain the infinite point with vanishing boundary condition $\ln\mathcal{E} \underset{x,\rho\to\infty}{\to} 0$), the general solution of (5.5) in this domain may be represented by the contour integral over boundary ∂D (which is nothing but axisymmetric version of familiar representation of arbitrary static electric field inside of a shell as an induced field of certain charge distribution on the shell):

$$\ln\mathcal{E} = \int_{\partial D} \frac{f(\kappa)d\kappa}{\{(\kappa-\xi)(\kappa-\bar{\xi})\}^{1/2}}\ , \tag{5.6}$$

with arbitrary function $f(\kappa)$ defined on ∂D and satisfying reality condition

$$f(\bar{\kappa}) = \overline{f(\kappa)}\ .$$

Solution (5.6) may be represented in the form (5.4), where dW is some locally holomorphic 1-form on the rational curve $\mathcal{L}_0$ (4.11) with $g_0 = 0$. For that one should take

$$dW(\lambda) = \frac{1}{2}\oint_{\partial D} f(\kappa)(dW_\kappa(\lambda) - dW_{\kappa^*}(\lambda))d\kappa\ , \tag{5.7}$$

where $dW_\kappa(\lambda)$ is differential of the 2nd kind on $\mathcal{L}_0$ with unique simple pole at $\lambda = \kappa$ and the following local expansion at $\lambda = \kappa$:

$$dW_\kappa(\lambda) \underset{\lambda\to\kappa}{=} \left(\frac{1}{(\lambda-\kappa)^2} + O(1)\right) d\lambda\ . \tag{5.8}$$

In the present $g_0 = 0$ case we can write $dW_\kappa(\lambda)$ explicitly as follows:

$$dW_\kappa(\lambda) = \frac{\gamma'(\kappa)\gamma'(\lambda)d\lambda}{(\gamma(\lambda)-\gamma(\kappa))^2}\ , \tag{5.9}$$

and, therefore,

$$\int_{\infty^1}^{\infty^2} dW_\kappa = \partial_\kappa(\ln\gamma)|_{\gamma=\infty}^{\gamma=0} = \frac{1}{\{(\kappa-\xi)(\kappa-\bar{\xi})\}^{1/2}}\ , \tag{5.10}$$

which shows coincidence of (5.4) with (5.6).

The partial integration in (5.7) leads to the following alternative representation of dW:

$$dW(\lambda) = \oint_{\partial D} h(\kappa)dW_{\kappa\,\kappa^*}(\lambda)d\kappa\ , \tag{5.11}$$

where

$$h(\kappa) = -\frac{f'(\kappa)}{2} \tag{5.12}$$

and $dW_{\kappa\,\kappa^*}(\lambda)$ is the standard differential of the 3rd kind given by

$$dW_{\kappa\,\kappa^*}(\lambda) = \left(\frac{\gamma'(\lambda)}{\gamma(\lambda)-\gamma(\kappa)} - \frac{\gamma'(\lambda)}{\gamma(\lambda)-\gamma^{-1}(\kappa)}\right) d\lambda \; . \tag{5.13}$$

Integral representation (5.11) of dW is the counterpart of the partially integrated version of (5.6):

$$\ln \mathcal{E} = 2\int_{\partial D} h(\kappa) \ln \gamma(\kappa, \xi, \bar{\xi}) d\kappa \; .$$

Performing in (5.7) the partial integration in a different way we can represent dW as a contour integral over κ of meromorphic 1-forms having poles of arbitrary fixed order at $\lambda = \kappa$ and $\lambda = \kappa^*$. We see that from the local point of view the set of 1-forms dW allowed in (5.4) is over-complete, and in order to get general local static solution it is sufficient to restrict ourselves by forms dW represented by (5.11).

5.1.2 Forms dW for higher genus

Denote by $dW_{QR}(P)$ the differential of the 3rd kind on $\mathcal{L}_0$ with poles of the first order at Q and R and residues $+1$ and -1 respectively, satisfying normalization conditions

$$\oint_{a_j} dW_{QR}(P) = 0 \; , \qquad \forall j \; . \tag{5.14}$$

The vector of b-periods of dW_{QR}:

$$2\pi i (B_{QR})_j \equiv \oint_{b_j} dW_{QR}$$

may be expressed in the following way in terms of basic holomorphic 1-forms dV_j:

$$(B_{QR})_j = \int_R^Q dV_j \; . \tag{5.15}$$

For $g_0 \geq 1$ it takes place the situation similar to $g_0 = 0$: locally the same 1-form $dW(P \in \mathcal{L}_0)$ can be represented in many different ways as contour integral of elementary differentials. The representation which it will be convenient to use in this paper is a $g_0 \geq 1$ counterpart of (5.11):

$$dW = \oint_{\partial D} h(\kappa) dW_{\kappa\,\kappa^*} \, d\kappa \; . \tag{5.16}$$

For vector of b-periods we have

$$(B_W)_j = 2\oint_{\partial D} h(\kappa) V_j|_{\xi}^{\kappa} d\kappa \; . \tag{5.17}$$

Now one can easily see the link to papers [27], where it was noticed that certain combination of components of the function Ψ from (2.1) corresponding to solution (5.1), (5.16), solves scalar Riemann-Hilbert problem on curve $\mathcal{L}_0$ with contour ∂D. Partial integration in (5.16), (5.17) gives the following relation between our function $h(\kappa)$ and conjugation function of RH problem $G(\kappa)$ from [27]:

$$\partial_\kappa \ln G(\kappa) = -4\pi i h(\kappa) \; . \tag{5.18}$$

The proof of (5.18) can be obtained expressing corresponding objects in terms of the prime-form on $\mathcal{L}_0$.

Now we are going to discuss the relationship between solutions (4.26), which we derived exploiting the link between Ernst and Schlesinger equations, and previuosly known class of solutions of Ernst equation (5.1). Apparently, solutions (4.26) contain less parameters; however, it turns out that in some sense both classes of solutions coincide.

5.2 Solutions (4.26) from solutions (5.1)

To describe this link we shall briefly discuss how abelian differentials of the 1st kind on $\mathcal{L}_0$ arise as limits of abelian differentials of the 3rd kind.

5.2.1 Holomorphic 1-forms as limits of meromorhic 1-forms

Cut $\mathcal{L}$ along basic cycles started at the same point and consider $dW_{QR}(P)$ in the fundamental polygon $\hat{\mathcal{L}}_0$ whose boundary consists of the cycles a_j^{0+}, b_j^{0+}, a_j^{0-}, b_j^{0+}.

Obviously, in the limit $Q \to R$ differential dW_{QR} should turn into certain holomorphic 1-form. The most naive way to perform this limit is to take $Q \to R$ inside of $\hat{\mathcal{L}}_0$; then $dW_{QR} \to 0$ and $B_{QR} \to 0$. However, this limit may also be taken in two different non-trivial ways:

1. Let Q tends to some boundary point of $\hat{\mathcal{L}}_0$, say, to a point belonging to the contour b_j^{0+}; and let R tend simultaneously to corresponding point of the contour b_j^{0-}. Then the limiting procedure does not influence the normalization conditions (5.14), and, since in the limit dW_{QR} turns into holomorphic 1-form, this 1-form must vanish:

$$dW_{QR} \underset{Q\to R\in b_j^0}{\to} 0 \,. \tag{5.19}$$

   However, the vector of b-periods of dW_{QR} does not vanish even in the limit: from (5.15) we see that

$$(B_{QR})_k \underset{Q\to R\in b_j^0}{\to} -\delta_{jk} \,. \tag{5.20}$$

2. Alternatively, we can treat the limit assuming that Q tends to some point of $\hat{\mathcal{L}}_0$ on the contour a_j^{0+}, and R tends to the same point of $\mathcal{L}_0$ belonging to the contour a_j^{0-}. Since the poles of dW_{QR} meet exactly on the cycle a_j^0, we see that the integral $\oint_{a_j^0} dW_{QR}$ does not vanish in the limit any more. Since $\oint_{a_k^0} dW_{QR}$ still vanishes for all $k \neq j$, we conclude that in the limit dW_{QR} becomes proportional to dV_j. To find the coefficient of proportionality, we calculate vector of b-periods of dW_{QR} in the limit according to (5.15) and find that

$$B_{QR} \underset{Q\to R\in a_j^0}{\to} \mathbf{B}_{0j} \,, \tag{5.21}$$

   where $\mathbf{B}_{0j}$ stands for jth column of the matrix of b-periods. Therefore, for dW_{QR} itself we have

$$dW_{QR} \underset{Q\to R\in a_j^0}{\to} 2\pi i\, dV_j \,. \tag{5.22}$$

Now choose differential dW in solution (5.1) in the following form:

$$dW = \sum_{j=1}^{g_0} (r_j dW_{Q_j R_j} - s_j dW_{\tilde{Q}_j \tilde{R}_j}) \tag{5.23}$$

with some constant vectors $\mathbf{r}, \mathbf{s} \in \mathbb{C}^{g_0}$, and take the limits $Q_j \to R_j \in a_j^0$, $\tilde{Q}_j \to \tilde{R}_j \in b_j^0$ as explained in items 1 and 2 above. In this limit, according to (5.19), (5.20), (5.21) (5.22), we get

$$B_W \to \mathbf{B}_0 \mathbf{r} + \mathbf{s} \,, \tag{5.24}$$

$$W|_{\infty^2}^{\infty^1} \to \sum_{j=1}^{g_0} r_j V_j|_{\infty^2}^{\infty^1} \,. \tag{5.25}$$

According to reality conditions (5.2) we should impose the following restriction on constants $\mathbf{r}$ and $\mathbf{s}$:

$$\mathbf{B}_0 \mathbf{r} + \mathbf{s} \in i\mathbb{R}^{g_0} \,,$$

coinciding with (4.19). Therefore, solution (5.1) turns in this limit into

$$\mathcal{E} = \frac{\Theta\left(V|_{\xi}^{\infty^1} + \mathbf{B}_0 \mathbf{r} + \mathbf{s} \Big| \mathbf{B}_0\right)}{\Theta\left(V|_{\xi}^{\infty^2} + \mathbf{B}_0 \mathbf{r} + \mathbf{s} \Big| \mathbf{B}_0\right)} \exp\left\{\sum_{j=1}^{g_0} r_j V_j|_{\infty^2}^{\infty^1}\right\} ,$$

which coincides with (4.26) if we take into account that the theta-function with characteristics in nothing but the ordinary theta-function with shifted argument multiplied with certain exponential factor (3.9).

Next we shall consider more non-trivial procedure of coming from (4.26) to (5.1), (5.16).

6 General algebro-geometric solutions of Ernst equation as limits of Schlesinger-related ones

Here we shall describe the inverse procedure: how to get the class of solutions (5.1), (5.16) starting from solutions (4.26).

6.1 Partial degeneration of spectral curve

Let us consider solution (4.26) with curve $\mathcal{L}_0$ substituted by the curve $\mathcal{L}_1$ of genus $g_1 = g_0 + n$ defined by the equation

$$\nu^2 = (\lambda - \xi)(\lambda - \bar{\xi}) \prod_{j=1}^{2g_0+2n} (\lambda - \lambda_j) \,, \tag{6.1}$$

and choose the vectors $\mathbf{r}$, $\mathbf{s} \in \mathbb{C}^{g_0+n}$ in the following way:

$$\mathbf{s} = 0 \,; \quad r_j = 0 \,, \quad 1 \le j \le g_0 \,; \quad r_{j+g_0} = h_j \in \mathbb{R} \,, \quad 1 \le j \le n \,. \tag{6.2}$$

Without loss of generality we shall assume $h_j \in [0,1]$. Denote by $\mathbf{B}_1$ the matrix of b-periods of $\mathcal{L}_1$ and by $dV_1, \ldots, dV_{g_0+n}$ the basis of normalized holomorphic 1-forms on $\mathcal{L}_1$. Now consider the solution (4.26) constructed from these data:

$$\mathcal{E} = \frac{\Theta \left[\begin{smallmatrix}\mathbf{r}\\0\end{smallmatrix}\right] \left(V|_\xi^{\infty^1} \Big| \mathbf{B}_1 \right)}{\Theta \left[\begin{smallmatrix}\mathbf{r}\\0\end{smallmatrix}\right] \left(V|_\xi^{\infty^2} \Big| \mathbf{B}_1 \right)} \,, \tag{6.3}$$

and take the limit

$$\lambda_{2g_0+2j+1} \,, \lambda_{2g_0+2j+2} \to \kappa_j \in \mathbb{R} \,. \tag{6.4}$$

Then curve $\mathcal{L}_1$ turns into $\mathcal{L}_0$ with double points at κ_j , $j = 1, \ldots, n$. The basis of holomorphic 1-forms of $\mathcal{L}_1$ turns into

$$dV_1, \ldots, dV_{g_0}, \frac{1}{2\pi i} dW_{\kappa_1 \kappa_1^*}, \ldots, \frac{1}{2\pi i} dW_{\kappa_n \kappa_n^*} \,,$$

where $dV_1, \ldots, dV_{g_0}$ is the basis of normalized holomorphic 1-forms on $\mathcal{L}_0$, and $dW_{\kappa_j \kappa_j^*}$ are normalized 1-forms of the 3rd kind on $\mathcal{L}_0$ with simple poles at κ_j and κ_j^* and residues $+1$ and -1 respectively. Therefore, the matrix of b-periods of $\mathcal{L}_1$ in the limit (6.4) may be described in terms of the objects associated to the curve $\mathcal{L}_0$ as follows:

$$(\mathbf{B}_1)_{jk} = (\mathbf{B}_0)_{jk} + o(1) \,, \qquad 1 \le j, k \le g_0 \,; \tag{6.5}$$

$$(\mathbf{B}_1)_{j\; k+g_0} = 2 V_j|_\xi^{\kappa_k} + o(1) \,, \qquad 1 \le j \le g_0 \,, \qquad 1 \le k \le n \,; \tag{6.6}$$

$$(\mathbf{B}_1)_{j+g_0\; j+g_0} = -\frac{1}{\pi i} \ln |\lambda_{2g_0+2j+1} - \lambda_{2g_0+2j+2}| + O(1) \,, \qquad 1 \le j \le n \,; \tag{6.7}$$

$$(\mathbf{B}_1)_{k+g_0\; j+g_0} = O(1) \,, \qquad 1 \le j \ne k \le n \,.$$

Substituting these relations in the definition of theta-function we can express the value of Ernst potential (6.3) in the limit (6.4) in terms of the objects associated to the curve $\mathcal{L}_0$:

$$\mathcal{E} = \frac{\Theta \left(V|_\xi^{\infty^1} + 2 \sum_{j=1}^n h_j V|_\xi^{\kappa_j} \Big| \mathbf{B}_0 \right)}{\Theta \left(V|_\xi^{\infty^2} + 2 \sum_{j=1}^n h_j V|_\xi^{\kappa_j} \Big| \mathbf{B}_0 \right)} \exp \left\{ \sum_{j=1}^n h_j W_{\kappa_j \kappa_j^*}|_{\infty^2}^{\infty^1} \right\} \tag{6.8}$$

The formula (4.28) for the metric coefficient F transforms into

$$F = \frac{2}{\Re \mathcal{E}} \Im \Big\{ \sum_{j=1}^n (\mathcal{A}_0^{-1})_{g_0 j} \frac{\partial}{\partial z_j} \ln \Theta \Big(V|_\xi^{\infty^2} + 2 \sum_{j=1}^n h_j V|_\xi^{\kappa_j} \Big| \mathbf{B}_0 \Big) + \sum_{j=1}^n h_j \frac{dW_{\kappa_j \kappa_j^*}}{d\lambda^{-1}} (\infty^2) \Big\} \tag{6.9}$$

For coefficient e^{2k} we get from (4.31):

$$e^{2k} = \{\det \mathcal{A}_0\}^{-\frac{1}{2}} \prod_{j=1}^{2g_0} |\lambda_j - \xi|^{-\frac{1}{4}} \Theta \Big(\sum_{j=1}^n h_j (U|_1^{\gamma(\kappa_j)} - \mathbf{e}_1) \Big| \mathbf{B} \Big)$$

$$\times \exp \Big\{ 2\pi i \sum_{j=1}^n h_j^2 \beta_j + \pi i \mathbf{B}_{11} \Big(\sum_{j=1}^n h_j \Big)^2 + \sum_{j \ne k,\, j,k=1}^n h_j h_k W_{\kappa_k \kappa_k^*}|_{\kappa_j^*}^{\kappa_j} \Big\} ; , \tag{6.10}$$

where as before $\mathbf{B}$ is $2g_0 - 1 \times 2g_0 - 1$ matrix of b-periods of curve $\mathcal{L}$; β_j is the second term of asymptotical expansion of $(\mathbf{B}_1)_{g+j\; g+j}$ as $\lambda_{g+2j+1} \to \lambda_{g+2j+2}$ $(j = 1, \ldots, n)$:

$$(\mathbf{B}_1)_{g+j\; g+j} = \frac{1}{\pi i} \ln |\lambda_{g+2j+1} - \lambda_{g+2j+2}| + \beta_j + o(1) \,.$$

Remark 6.1 The assumption $\kappa_j \in \mathbb{R}$, $h_j \in \mathbb{R}$ was only made to shorten the presentation; one can also allow in (6.8) the presence of conjugated pairs $\kappa_j = \bar{\kappa}_l$, $h_j = \bar{h}_l$ which come out if we glue together two "vertical" branch cuts.

6.2 Continuous limit: condensation of double points

Now we can take a continuous limit in (6.8) distributing points κ_j over an arbitrary contour ∂D with an arbitrary (say, continuous) measure $h(\kappa)$ satisfying reality condition

$$h(\bar{\kappa}) = \overline{h(\kappa)} \ .$$

Then (6.8) turns into

$$\mathcal{E} = \frac{\Theta\left(V|_{\xi}^{\infty^1} + 2\oint_{\partial D} h(\kappa) V|_{\xi}^{\kappa} d\kappa \Big| \mathbf{B}_0\right)}{\Theta\left(V|_{\xi}^{\infty^2} + 2\oint_{\partial D} h(\kappa) V|_{\xi}^{\kappa} d\kappa \Big| \mathbf{B}_0\right)} \exp\left\{\oint_{\partial D} h(\kappa) W_{\kappa\kappa^*}|_{\infty^2}^{\infty^1} d\kappa\right\} \tag{6.11}$$

coinciding with (5.1), (5.16), (5.17).

Taking continuous limit in the formulas (6.9), (6.10), we come to the following expressions for the metric coefficients e^{2k} and F corresponding to the Ernst potential (6.11).

Theorem 6.1 *Coefficient F of the metric (2.12), corresponding to solution (6.11) of the Ernst equation, is given by*

$$F =$$

$$\frac{2}{\Re \mathcal{E}} \Im \Big\{ \sum_{j=1}^{g_0} (\mathcal{A}_0^{-1})_{g_0 j} \frac{\partial}{\partial z_j} \ln \Theta \left(V|_{\xi}^{\infty^2} + 2 \oint_{\partial D} h(\kappa) V|_{\xi}^{\kappa} d\kappa \Big| \mathbf{B}_0 \right) + \oint_{\partial D} h(\kappa) d\kappa \frac{dW_{\kappa\kappa^*}}{d\lambda^{-1}}(\infty^2) \Big\} \ , \tag{6.12}$$

where all the objects are associated to curve $\mathcal{L}_0$. Metric coefficient e^{2k} is, up to an arbitrary constant factor, given by the following expression:

$$e^{2k} = \{\det \mathcal{A}_0\}^{-\frac{1}{2}} \prod_{j=1}^{2g_0} |\lambda_j - \xi|^{-\frac{1}{4}} \Theta\left(\oint_{\partial D} h(\kappa) (U|_1^{\gamma(\kappa)} - \mathbf{e}_1) d\kappa \Big| \mathbf{B} \right)$$

$$\times \exp\left\{ \pi i \mathbf{B}_{11} \Big(\oint_{\partial D} h(\kappa) d\kappa \Big)^2 + \oint_{\partial D} \oint_{\partial D} \left(W_{\tilde{\kappa}\tilde{\kappa}^*}|_{\kappa^*}^{\kappa} - 2 \ln|\kappa - \tilde{\kappa}| \right) d\kappa d\tilde{\kappa} \right\} \ , \tag{6.13}$$

where $\mathbf{B}$ is the matrix of b-periods and dU is the normalized basis of holomorphic 1-forms on curve $\mathcal{L}$.

Proof. Formulas (6.12) and (6.13) are direct continuous analogs of (6.9) and (6.10), respectively. Term $2\ln|\kappa - \tilde{\kappa}|$ is subtracted, using the freedom to renormalize e^{2k} with an arbitrary $(\xi, \bar{\xi})$-independent factor, to achieve convergence of the double integral at $\kappa = \tilde{\kappa}$.

References

[1] S.V.Kovalevski, Sur le problème de la rotation d'un corps solide autour d'un point fixe, Acta Math. **12** 177-232 (1889)

[2] Dobriner

[3] C.Neumann, De problemate quodam mechanico, quod ad primam integralium ultraellipticorum classem revocatur, J.Reine Angew.Math. **56** (1859), 48-50

[4] Floke

[5] Lyapunov

[6] Glazman

[7] J.Drach, Compt.Rend.Acad.Sci. **168** (7), 337-340 (1919)

[8] C.S.Gardner, J.M.Green, M.D.Kruskal, R.M.Miura, Method for solving Korteveg-deVries equation, Phys.Rev.Lett. **19** 1095-1098 (1967)

[9] A.R.Its, V.B.Matveev, Schrödinger operators with finite-gap spectrum and N-soliton solutions of Korteveg-deVries equation, Theor.Mat.Phys. **23** (1), 51-67 (1975)

[10] A.R.Its, V.P.Kotlyarov, About one class of solutions of non-linear Schrödinger equation, Dokl.Ukr.Akad.Nauk, ser. A, **11** (1976) 965-968

[11] V.A.Kozel, V.P.Kotlyarov, Almost periodic solutions of the equation $u_{tt} - u_{xx} + \sin u = 0$, Dokl.Ukr.Akad.Nauk, ser. A **10**, 878-881 (1976)

[12] I.M.Krichever, Algebraic curves and non-linear difference equations, Russ.Math.Surv. **33** No.4, 255-256 (1978)

[13] I.M.Krichever, Methods of algebraic geometry in the theory of non-linear equations, Uspekhi Matem. Nauk, **32** No.6 (1977) 183-208

[14] A.I.Bobenko, All constant mean curvature tori in $\mathbf{R}^3, \mathbf{S}^3, \mathbf{H}^3$ in terms of theta-functions, Math.Ann. **290** 209-245 (1991)

[15] T.Shiota, Characterization of Jacibian varieties in terms of soliton equations, Invent. Math. **83** (1986) 333-382

[16] M.F.Atiyah, N.J.Hitchin, The Geometry and Dynamics of Magnetic Monopoles, Princeton Univ. Press, Princeton, 1988

[17] V.A.Belinskii, V.E.Zakharov, Integration of the Einstein equations by the methods of inverse scattering theory and construction of explicit multisoliton solutions, JETP **48** (1978) 985

[18] D. Maison, Are the stationary, axially symmetric Einstein equations completely integrable?, Phys. Rev. Lett. **41** (1978) 521

[19] L.Bianchi, Lezioni di geometria differenziale, Pisa, 1909

[20] D. Korotkin, Finite-gap solutions of stationary axially symmetric Einstein equations in vacuum, *Theor.Math.Phys.* **77** 1018 (1989) (translated from Russian)

[21] D. Korotkin, V. Matveev, Algebro-geometrical solutions of gravitational equations, *St.Petersburg Math. J.* **1** 379-408 (1990) (translated from Russian)

[22] D. A. Korotkin, Elliptic solutions of stationary axisymmetric Einstein equation, *Class. Quantum Gravity*, **10** (1993), 2587-2613.

[23] G.Neugebauer, R.Meinel, General relativistic gravitational field of the rigidly rotating disk of dust: Solution in terms of ultraelliptic functions, Phys.Rev.Lett. 75 (1995) 3046-3048

[24] G.Neugebauer, R.Meinel, Solutions to Einstein's field equation related to Jacobi inversion problem, Phys.Lett A 210 (1996) 160

[25] D.Korotkin, Some remarks on finite-gap solutions of Ernst equation, Phys.lett.A **229** (1997) 195-199

[26] C.Klein, O.Richter, On a class of physically realistic solutions to the Ernst equation, Phys.Rev.Lett., **79**, 565 (1997)

[27] C.Klein, O.Richter, Explicit solutions of Riemann-Hilbert problem for the Ernst equation, Phys.Rev.D **57** 857-862

[28] L. Schlesinger, Über eine Klasse von Differentialsystemen beliebiger Ordnung mit festen kritischen Punkten, *J. Reine Angew. Math.* **141** (1912), 96-145.

[29] A.Kitaev, D.Korotkin, On solutions of the Schlesinger equations in terms of theta-functions, Preprint AEI-052, February, 1998, Intern.Math.Research Notices (19980, to appear

[30] D. Korotkin, H. Nicolai, Isomonodromic quantization of dimensionally-reduced gravity, Nucl.Phys.B **475** (1996) 397-439

[31] John D. Fay, Theta-functions on Riemann surfaces, Lect.Notes Math. v.352, Springer, 1973

[32] M. Jimbo, T. Miwa, and K. Ueno, Monodromy preserving deformation of linear ordinary differential equations with rational coefficients I, *Physica* **2D** (1981), 306-352.